EXPLICIT ESTIMATES IN THE THEORY OF

DISTRIBUTION OF PRIMES

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "EXPLICIT ESTIMATES IN THE THEORY OF DISTRIBUTION OF PRIMES", submitted by DENIS HANSON in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis is primarily concerned with two problems of the theory of distribution of primes.

In Chapter I an upper bound for the product of the primes not exceeding n is obtained by elementary means. If we denote by $B(n)$ the least common multiple of the integers $1, 2, \ldots, n$, then it is shown that

$$B(n) < 3^n, \qquad n \geq 0$$

Chapter I concludes with two miscellaneous results on the distribution of primes.

In Chapter II an elementary approach is applied to obtain a refinement of a theorem of Sylvester and Schur related to the prime divisors of the product of consecutive integers.

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CHAPTER I

ON THE PRODUCT OF THE PRIMES

§1.1 The least common multiple of the integer 1, 2, ..., n.

In recent years several attempts have been made to obtain estimates of an upper bound for the product of the primes less than or equal to a given integer n. Denote by $A(n) = \prod_{p \leq n} p$ the above mentioned product and define as usual

$$\vartheta(n) = \sum_{p \leq n} \log p = \log \prod_{p \leq n} p$$

$$\psi(n) = \sum_{p^\alpha \leq n} \log p$$

Analysis of binomial and multinomial coefficients has led to results such as $A(n) < 4^n$, due to Erdös and Kalmar [4]. A note by L. Moser [8] gave an inductive proof of $A(n) < (3.37)^n$. More accurate results are known, in particular those in a paper of Rosser and Schoenfeld [14] in which they prove $\vartheta(n) < 1.01624n$; however their methods are considerably deeper and involve complex variable theory as well as heavy computations. Using only elementary methods we will prove the following theorem, which improves the results of [4] and [8] considerably.

<u>Theorem 1.1.1</u> Let $B(n)$ denote the least common multiple of the numbers 1, 2, ..., n, then $B(n) < 3^n$.

We might note that if for a given prime p, α_p is such that p^{α_p} is the highest power of p not exceeding n, then $B(n)$ is the product of the p^{α_p} taken over all the primes $p \leq n$, so that

$$B(n) = \prod_{p \leq n} p^{\alpha_p} \quad \text{or} \quad B(n) = \prod_{p^{\alpha} \leq n} p.$$

<u>Lemma 1.1.1</u> If $a_1, \ldots, a_k$ are positive integers such that $\sum_{i=1}^{k} \frac{1}{a_i} \leq 1$ and if $a_k > x \geq 1$, for x real, then

$$(1.1.1) \qquad [x] > \sum_{i=1}^{k} \left[\frac{x}{a_i}\right]$$

where the square brackets denote the greatest integer function.

<u>Proof</u>: First, for a and b real, say $a = n + \nu$ and $b = m + \mu$ where m and n are non-negative integers and $0 < \nu, \mu < 1$, we have

$$[a] + [b] = m + n \leq [m + \mu + n + \nu] = [a + b];$$

hence,

$$\sum_{i=1}^{k} \left[\frac{x}{a_i}\right] = \sum_{i=1}^{k-1} \left[\frac{x}{a_i}\right] \leq \left[\sum_{i=1}^{k-1} \frac{x}{a_i}\right] \leq \left[x - \frac{x}{a_k}\right]$$

and therefore

(1.1.2) $$\sum_{i=1}^{k} \left[\frac{x}{a_i}\right] < [x] \quad \text{if } x \text{ is an integer; however,}$$

since the a_i, $i = 1, \ldots, k$, are positive integers,

(1.1.3) $$\left[\frac{x}{a_i}\right] = \left[\frac{[x]}{a_i}\right] \qquad \text{(see [10])}$$

for, if we write $x = n + \nu$, $0 \leq \nu < 1$, n an integer then $n = q\,a_i + r$ where $0 \leq r \leq a_i - 1$, q an integer ≥ 0, and it follows that

(1.1.4) $$\left[\frac{x}{a_i}\right] = \left[\frac{qa_i + r + \nu}{a_i}\right] = q + \left[\frac{r + \nu}{a_i}\right] = q$$

since $0 \leq r + \nu < a_i$.

On the other hand we have

(1.1.5) $$\left[\frac{[x]}{a_i}\right] = \left[\frac{n}{a_i}\right] = \left[q + \frac{r}{a_i}\right] = q$$ which then together

with (1.1.4) implies (1.1.3).

Therefore since the lemma holds for x an integer, (1.1.3) implies the lemma is valid for all $x \geq 1$.

In particular, let us now consider the set of a_i's defined by

$$a_1 = 2, \quad a_{n+1} = a_1 a_2 \ldots a_n + 1$$

A simple inductive proof shows that the a_i defined in this manner satisfy the following recurrence relation, $a_1 = 2$, $a_{n+1} = a_n^2 - a_n + 1$ and clearly the conditions of lemma 1.1.1 are satisfied by the a_i.

Define

(1.1.6) $$C(n) = \frac{n!}{\left[\frac{n}{a_1}\right]! \left[\frac{n}{a_2}\right]! \left[\frac{n}{a_3}\right]! \ldots}$$

where the a_i are as above. $C(n)$ may be seen to be an integer upon comparison with an appropriate multinomial coefficient.

Lemma 1.1.2 (Legendre) The exponent of a prime p in the prime power factorization of $n\,!$, where n is a natural number, is

$$(1.1.7) \qquad \alpha = \alpha(n,p) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots \qquad \text{(see [4])}$$

Proof: The numbers 1, 2, ..., n include just $\left[\frac{n}{p}\right]$ multiples of p, just $\left[\frac{n}{p^2}\right]$ multiples of p^2 and so on; hence since $n\,!$ is the product of the numbers 1, 2, ..., n, we have (1.1.7) .

Lemma 1.1.3 Any prime p such that $p \le n$, divides $C(n)$ to at least the α_p power where $p^{\alpha_p} \le n < p^{\alpha_p+1}$.

Proof: Given a prime $p \le n$, the power β_p to which it occurs in $C(n)$ is, by lemma 1.1.2,

$$(1.1.8) \qquad \beta_p = \sum_{i=1}^{[\log_p n]} \left(\left[\frac{n}{p^i}\right] - \left[\frac{n}{a_1 p^i}\right] - \left[\frac{n}{a_2 p^i}\right] - \cdots \right) .$$

Thus by choosing x to be n/p^i in lemma 1.1.1 we have, since $i \le [\log_p n]$, that $\frac{n}{p^i} \ge 1$ and

$$\left[\frac{n}{p^i}\right] > \sum_{j=1}^{\infty} \left[\frac{x}{a_j}\right]$$

and therefore

$$\left[\frac{n}{p^i}\right] - \left[\frac{n}{a_1 p^i}\right] - \left[\frac{n}{a_2 p^i}\right] - \dots \geq 1 \quad \text{for each } i . \tag{1.1.9}$$

Now since $[\log_p n]$ is such that

$$p^{[\log_p n]} \leq n < p^{[\log_p n] + 1}$$

we have that $\beta_p \geq \alpha_p$ where $p^{\alpha_p} \leq n < p^{\alpha_p + 1}$.

<u>Lemma 1.1.4</u>

$$\frac{\left(\frac{n}{a_i}\right)^{\frac{n}{a_i}}}{\left[\frac{n}{a_i}\right]^{\left[\frac{n}{a_1}\right]}} < \left(e\,\frac{n}{a_i}\right)^{\frac{a_i - 1}{a_i}} , \qquad n \geq a_i \quad .$$

<u>Proof</u>: If $n = a_i$ the result is trivial. If $n > a_i$ we have

$$\frac{\left(\frac{n}{a_i}\right)^{\frac{n}{a_i}}}{\left[\frac{n}{a_i}\right]^{\left[\frac{n}{a_i}\right]}} < \frac{\left(\frac{n}{a_i}\right)^{\frac{n}{a_i}}}{\left(\frac{n - a_i + 1}{a_i}\right)^{\frac{n - a_i + 1}{a_i}}}$$

$$= \frac{\left(\frac{n}{a_i}\right)^{\frac{n-a_i+1}{a_i}} \left(\frac{n}{a_i}\right)^{\frac{a_i-1}{a_i}}}{\left(\frac{n-a_i+1}{a_i}\right)^{\frac{n-a_i+1}{a_i}}} = \left(\frac{n}{n-a_i+1}\right)^{\frac{n-a_i+1}{a_i}} \left(\frac{n}{a_i}\right)^{\frac{a_i-1}{a_i}}$$

$$= \left(\left(1 + \frac{1}{\frac{n-a_i+1}{a_i-1}}\right)^{\frac{n-a_i+1}{a_i-1}} \right)$$

$$< \left(\frac{e\ n}{a_i}\right)^{\frac{a_i-1}{a_i}}$$

since $\log\left(1 + \frac{1}{x}\right)^x = x \log\left(1 + \frac{1}{x}\right)$

$$= x \int_1^{1+\frac{1}{x}} \frac{dt}{t} < x \cdot \frac{1}{x} \cdot 1 = 1$$

which implies $\left(1 + \frac{1}{x}\right)^x < e$.

We will now proceed to obtain an upper bound for $C(n)$ using the preceeding lemmas.

Lemma 1.1.5

$$C(n) < \frac{n^n}{\left[\frac{n}{a_1}\right]^{\left[\frac{n}{a_1}\right]} \left[\frac{n}{a_2}\right]^{\left[\frac{n}{a_2}\right]} \cdots \left[\frac{n}{a_k}\right]^{\left[\frac{n}{a_k}\right]}}$$

for a particular $k = k(n)$.

Proof: If $n = n_1 + n_2 + \ldots + n_k$ (where n and all the $n_i (i=1,\ldots,k)$ are integers,) then by the multinomial theorem we know that

$$(1.1.10) \qquad (n_1+n_2+\ldots+n_k)^n > \binom{n}{n_1,\ldots,n_k} n_1^{n_1} \ldots n_k^{n_k}$$

since the right hand side of (1.1.10) is just one term in the expansion of $(n_1+n_2+\ldots+n_k)^n$. Therefore if

$$n = \sum_{i=1}^{k} \left[\frac{n}{a_i}\right]$$ for some appropriate choice of k, the

lemma follows immediately from (1.1.10).

On the other hand, if

$$\sum_{i=0}^{\infty} \left[\frac{n}{a_i}\right] = t < n ,$$

$$C(n) = \frac{n(n-1)\ldots(t+1)\, t\,!}{\left[\frac{n}{a_1}\right]!\left[\frac{n}{a_2}\right]!\ldots\left[\frac{n}{a_k}\right]!} < \frac{n^{n-t}\, t^t}{\left[\frac{n}{a_1}\right]^{\left[\frac{n}{a_1}\right]}\left[\frac{n}{a_2}\right]^{\left[\frac{n}{a_2}\right]}\ldots\left[\frac{n}{a_k}\right]^{\left[\frac{n}{a_k}\right]}}$$

$$< \frac{n^n}{\left[\frac{n}{a_1}\right]^{\left[\frac{n}{a_1}\right]}\left[\frac{n}{a_2}\right]^{\left[\frac{n}{a_2}\right]}\ldots\left[\frac{n}{a_k}\right]^{\left[\frac{n}{a_k}\right]}} .$$

The magnitude of k satisfies the following:

Lemma 1.1.6 If $a_k \leq n < a_{k+1}$, then

(1.1.11) $$k < \log_2 \log_2 n+2 \quad \text{for} \quad k \geq 3.$$

Proof: We know $a_{k+1} = a_k^2 - a_k + 1 > (a_k - 1)^2 + 1$,

$$a_3 = 7 > 2^{2^1} + 1 .$$

Assume $a_k > 2^{2^{k-2}} + 1$ for $k > 3$;

then $$a_{k+1} > \left(2^{2^{k-2}}\right)^2 + 1 = 2^{2^{k-1}} + 1$$

and therefore

$$k < \log_2 \log_2 (a_k - 1) + 2 < \log_2 \log_2 n + 2 .$$

Finally, applying lemmas 1.1.4, 1.1.5 and 1.1.6 we have, if k is such that $a_k \leq n < a_{k+1}$,

$$(1.1.12)\quad C(n) < \frac{n^n\left(e\frac{n}{a_1}\right)^{\frac{a_1-1}{a_1}}\left(e\frac{n}{a_2}\right)^{\frac{a_2-1}{a_2}}\cdots\left(e\frac{n}{a_3}\right)^{\frac{a_k-1}{a_k}}}{\left(\frac{n}{a_1}\right)^{\frac{n}{a_1}}\left(\frac{n}{a_2}\right)^{\frac{n}{a_2}}\left(\frac{n}{a_3}\right)^{\frac{n}{a_3}}\cdots}$$

since $\left[\frac{n}{a_t}\right] = 0$ and $\left[\frac{n}{a_t}\right]! = 1$ for all $t > k$ and hence

$$\frac{1}{\left(\frac{n}{a_t}\right)^{\left(\frac{n}{a_t}\right)}} > 1 \quad \text{for} \quad t > k .$$

We observe that the product $a_1^{\frac{1}{a_1}} a_2^{\frac{1}{a_2}} \cdots a_k^{\frac{1}{a_k}}$ is monotonic increasing with k. Since a check of log tables reveals

$$\sum_{i=1}^{i=5} \log\left(a_i^{\frac{1}{a_i}}\right) < 1.08240 \quad \text{and} \quad \log\left(a_6^{\frac{1}{a_6}}\right) < 5 \times 10^{-6} \quad \text{and}$$

since we know $a_{i+1} = a_i^2 - a_i + 1$, then

$$a_i^2 > a_{i+1} > (a_i-1)^2 \qquad \text{for} \quad i \geq 1$$

and therefore it follows that

$$\frac{\log\left(a_{i+1}^{\frac{1}{a_{i+1}}}\right)}{\log\left(a_i^{\frac{1}{a_i}}\right)} = \frac{a_i \log a_{i+1}}{a_{i+1} \log a_i}$$

$$< \frac{2a_i}{a_{i+1}} < \frac{2a_i}{(a_i-1)^2} < \frac{1}{2} \quad \text{for} \quad i \geq 3$$

and therefore

$$\sum_{i=6}^{\infty} \log \left(a_i^{\frac{1}{a_i}} \right) < 5 \times 10^{-6} \left(\sum_{j=0}^{\infty} \frac{1}{2^j} \right) = 10^{-5} ,$$

which then implies $\sum_{i=1}^{\infty} \log a_i^{\frac{1}{a_i}} < 1.08241$ which implies, if we define

$$w = \lim_{k \to \infty} \left(a_1^{\frac{1}{a_1}} a_2^{\frac{1}{a_2}} \cdots a_k^{\frac{1}{a_k}} \right) ,$$

that $w < 2.952$.

Since

$$\frac{a_1-1}{a_1} + \frac{a_2-1}{a_2} + \cdots + \frac{a_k-1}{a_k} = \left(1 - \frac{1}{a_1}\right) + \left(1 - \frac{1}{a_2}\right) + \cdots + \left(1 - \frac{1}{a_k}\right)$$

$$= k - 1 + \frac{1}{a_{k+1}+1} ,$$

it follows from (1.1.12) that

$$\text{(1.1.13)} \qquad C(n) < \frac{(en)^{k-1+\frac{1}{a_{k+1}+1}} \, w^n}{a_1^{\frac{a_1-1}{a_1}} \, a_2^{\frac{a_2-1}{a_2}} \cdots a_k^{\frac{a_k-1}{a_k}}}$$

$$< e^{k-3/2} \, n^{k-3/2} \, w^n , \quad k > 2 \quad (\text{since } n \le a_1 a_2 \ldots a_k),$$

upon which a check of log tables reveals $C(n) < 3^n$ for $n > 1300$.

A check of tables of the function

$$\psi(n) = \sum_{p^\alpha \le n} \log p$$

(such as those of Appel and Rosser [1]) for $n \leq 1300$ concludes proof of theorem 1.1.1. Theorem 1.1.1 then states

$$\psi(n) < 1.09861\,n, \quad n > 0$$

compared with the result obtained by the analytic methods of Rosser and Schoenfeld [12]

i.e. $\psi(n) < 1.03883n, \quad n > 0.$

Obtaining a lower bound for the product of the primes by similar methods leads to a less elegant result for small n. The above mentioned paper of Rosser and Schoenfeld includes the results

$$\vartheta(x) = \sum_{p \leq x} \log p > .84x \quad \text{for} \quad x \geq 101$$

and $\vartheta(x) > .98x \quad \text{for} \quad x \geq 7481$.

We now prove the weaker theorem:

<u>Theorem 1.1.2</u> $\vartheta(n) = \sum_{p \leq n} \log p > (3/4)n , \quad n \geq 13$.

Define $D(n) = \dfrac{n!}{\left[\frac{n}{2}\right]!\left[\frac{n}{3}\right]!\left[\frac{n}{6}\right]!}$

where again the square brackets denote the greatest integer function.

Clearly, for $n > 1$, $D(n) \leq D(n-1)$ if and only if $n \equiv 0(6)$.

(1.1.14) <u>Lemma 1.1.7</u> $D(n) > \dfrac{(2^4\, 3^3)^{\frac{n}{6}}}{n^2} , \quad n \geq 2$.

<u>Proof</u>: First consider the case when $n \equiv 0(6)$, say $n = 6k$. We will proceed by induction:

$$D(6) = \binom{6}{3,\ 2,\ 1} = 60 > \frac{2^4\ 3^3}{6^2} = 12 .$$

Assume (1.1.14) to hold for $n = t > 1$

then $D(6(t+1)) = \dfrac{(6t+6)\ \ldots\ (6t+1)}{(3t+3)\ldots(3t+1)(2t+2)(2t+1)(t+1)} \cdot D(6t)$

hence

$$D(6(t+1)) = \frac{2^2\ 3\ (6t+5)(6t+1)}{(t+1)^2}\ D(6t)$$

$$> \frac{2^2\ 3\ (6t+5)(6t+1)}{(t+1)^2}\ \frac{(2^4\ 3^3)^t}{(6t)^2} > \frac{(2^4\ 3^3)^{t+1}}{6^2(t+1)^2}$$

since $2^2\ 3\ (6t+5)(6t+1) > 2^4\ 3^3\ t$ for $t \geq 1$ and thus lemma 1.1.7 hold for $n \equiv 0(6)$. We now want to show (1.1.14) holds in general. Consider the following cases:

(i) If $n \equiv 1\ (6)$, say $n = 6m+1$ then

$$D(6m+1) = (6m+1)\ D(6m) > (6m+1)\ \frac{(2^4\ 3^3)^m}{(6m)^2}$$

$$> \frac{(2^4\ 3^3)^{\frac{6m+1}{6}}}{(6m+1)^2}\ , \qquad m \geq 1 .$$

(ii) If $n \equiv 2(6)$, say $n = 6m+2$, then

$$D(6m+2) = 2(6m+1)\ D(6m) > \frac{(2^4\ 3^3)^{\frac{6m+2}{6}}}{(6m+2)^2}\ , \qquad m \geq 0 .$$

(iii) If $n \equiv 3(6)$, say $n = 6m+3$, then

$$D(6m+3) = 3.2.(6m+1) > \frac{(2^4\ 3^3)^{\frac{6m+3}{6}}}{(6m+3)^2}, \quad m \geq 0 .$$

(iv) If $n \equiv 4(6)$, say $n = 6m+4$, then

$$D(6m+4) = 2.3.2.(6m+1)\ D(6m) > \frac{(2^4\ 3^3)^{\frac{6m+4}{6}}}{(6m+4)^2}, \quad m \geq 0 .$$

And finally

(v) If $n \equiv 5(6)$, say $n = 6m+5$,

$$D(6m+5) = (6m+5)\ 2.3.2.(6m+1)\ D(6m) > \frac{(2^4\ 3^3)^{\frac{6m+5}{6}}}{(6m+5)^2}, \quad m \geq 0 ,$$

which proves lemma 1.1.7 .

A prime p occurs in $D(n)$ to the exponent α_p given by

$$(1.1.16) \qquad \alpha_p = \sum_{i=1}^{[\log_p n]} \left(\left[\frac{n}{p^i}\right] - \left[\frac{n}{2p^i}\right] - \left[\frac{n}{3p^i}\right] - \left[\frac{n}{6p^i}\right] \right) .$$

Lemma 1.1.8 If p divides $D(n)$ to a power α_p then

$p^{\alpha_p} \leq n^2$.

Proof: Consider $d_{n,p^i} = \left[\frac{n}{p^i}\right] - \left[\frac{n}{2p^i}\right] - \left[\frac{n}{3p^i}\right] - \left[\frac{n}{6p^i}\right]$.

We can express any n in the following manner:

$$n = \beta_{p,i}\, p^i + \alpha_{p,i} \quad \text{where} \quad 0 \leq \alpha_{p,i} < p^i .$$

If we now consider the cases

$$\beta_{p,i} \equiv j\ (6) \quad \text{where} \quad j = 0,1,\ldots,5 \,,$$

we find $\beta_{p,i} = 0,1,1,1,1,2$ respectively since, for example,

$$\beta_{p,i} \equiv 5(6), \quad \text{i.e. if} \quad \beta_{p,i} = 6a+5 \quad \text{say, then}$$

$$d_{n,p^i} = \left[\frac{(6a+5)p^j+\alpha_{p,i}}{p^i}\right] - \left[\frac{(6a+5)p^i+\alpha_{p,i}}{2p^i}\right] - \left[\frac{(6a+5)p^i+\alpha_{p,i}}{3p^i}\right] - \left[\frac{(6a+5)p^i+\alpha_{p,i}}{6p^i}\right]$$

$$= \left[6a+5 + \frac{\alpha_{p,i}}{p^i}\right] - \left[3a+2 + \frac{\alpha_{p,i}+1}{2p^i}\right] - \left[2a+1 + \frac{\alpha_{p,i}+2}{3p^i}\right] - \left[a + \frac{\alpha_{p,i}+5}{6p^i}\right]$$

$$= 6a+5 - (3a+2) - (2a+1) - a = 2.$$

Therefore if δ_p is such that

$$p^{\delta_p} \leq n < p^{\delta_p+1} \,, \quad \text{then we have } \delta_p \text{ terms in the}$$

summation (1.1.16), and thus the maximum contribution of any prime p to $D(n)$ is n^2 since each term of (1.1.16) is 0, 1 or 2, which is lemma 1.1.8 .

Note that if $\alpha_p > 2$, the summation must be over at least two values of i, i.e. $p^2 \leq n$.

Lemma 1.1.9

$$D(n) = \frac{n\,!}{\left[\frac{n}{2}\right]!\left[\frac{n}{3}\right]!\left[\frac{n}{6}\right]!} < \prod_{p \leq n} p \prod_{p \leq \frac{n}{5}} p \; n^{n^{1/2}}$$

Proof: Consider the cases where p lies in the ranges:

(i) $n \geq p > \frac{n}{2}$

(ii) $\frac{n}{2} \geq p > \frac{n}{3}$

(iii) $\frac{n}{3} \geq p > \frac{n}{4}$

(iv) $\frac{n}{4} \geq p > \frac{n}{5}$.

In each of the four cases we have $\alpha_p = 1$ since in (i) p divides the numerator but not the denominator.
In (ii) p^2 divides the numerator while only p divides the denominator.
In (iii) p^3 divides the numerator while only p^2 divides the denominator.
In (iv) p^4 divides the numerator while only p^3 divides the denominator.
A similar check of the intervals

(v) $\frac{n}{5} \geq p > \frac{n}{6}$

(vi) $\frac{n}{6} \geq p > \frac{n}{7}$

shows that $\alpha_p = 2$ and $\alpha_p = 0$ respectively, hence since the primes $\leq \frac{n}{5}$ may occur twice and those $\leq n^{1/2}$ may contribute as much as n^2, we have

$$D(n) < \prod_{p \leq n} p \prod_{p \leq \frac{n}{5}} p \quad n^{2\,\pi(n^{1/2})} .$$

But clearly $\pi(n) \leq \frac{n}{2}$ for $n \geq 8$; therefore

$$D(n) < \prod_{p \leq n} p \prod_{p \leq \frac{n}{5}} p \quad n^{n^{1/2}} , \qquad n \geq 64 .$$

Thus by lemmas 1.1.7 and 1.1.9 we have

$$\frac{\left(2^4\ 3^3\right)^{\frac{n}{6}}}{n^2} < \prod_{p \leq n} p \quad 3^{\frac{n}{5}} \quad n^{n^{1/2}} , \qquad n \geq 64 ,$$

or $\vartheta(n) > .79169n - (2+n^{1/2}) \log n$

$$> \frac{3}{4} n \quad \text{for} \quad n \geq 8 \times 10^4,$$

and a simple check of tables of the $\vartheta(n)$ function (such as those of Appel and Rosser [1]) for $n < 8 \times 10^4$ concludes the proof of theorem 1.1.2 for $n > 13$.

§1.2 Miscellaneous results involving inequalities on $\vartheta(n)$ and $\pi(n)$.

(i). Since $\vartheta(n) < \psi(n)$, $n > 3$, we have

(1.2.1) $$\frac{3}{4} n < \vartheta(n) < \psi(n) < n \log 3$$

(1.2.1) implies immediately that there is a prime between $2n$ and $3n$, for

$$\vartheta(3n) > 2.25n > 2n \log 3 > \vartheta(2n).$$

The previously mentioned paper of Rosser and Schoenfeld [12] includes the bounds

(1.2.2) $$.98000n < \vartheta(n) < 1.01624n, \qquad n > 7481.$$

A paper by Rohrbach and Weis [11] shows there is a prime between

n and $\frac{14}{13}n$ for $n \geq 118$; however (1.2.2) obviously offers a better result since

(1.2.3) $$\vartheta(28n) > 27.4400n > 27.43848n > \vartheta(27n)$$

which implies the existence of a prime between $27n$ and $28n$, and a check of table of primes such as those of D. N. Lehmer [7] indicates n must be at **least** 3.

(ii). Rosser and Schoenfeld [12] prove **the result:**
if $\pi(x)$ denotes the number of primes less then or equal to x, then for $1 < x < 113$ and for $113.6 \leq x$

(1.2.4) $$\pi(x) < \frac{5x}{4 \log x}$$

and for $x = 113$,

(1.2.5) $$\pi(x) = 1.25506 \frac{x}{\log x}$$

By theorem 1.1.1 we know $\psi(x) = \sum_{p^m \leq x} \log p < x \log 3$;

thus by summation by parts we have

$$\pi(x) = \sum_{p \leq x} 1 = \sum_{n=2}^{x} \frac{\psi(n) - \psi(n-1)}{\log n}$$

$$= \frac{\psi(2) - \psi(1)}{\log 2} + \frac{\psi(3) - \psi(2)}{\log 3} + \dots + \frac{\psi(x) - \psi(x-1)}{\log x}$$

$$= \sum_{n=2}^{x} \psi(n) \left(\frac{1}{\log n} - \frac{1}{\log (n+1)}\right) + \frac{\psi(x)}{\log x} .$$

Therefore, since $n(\log (n+1) - \log n) = \log\left((1+\frac{1}{n})^n\right) < 1$,

(1.2.6) $$\pi(x) < \frac{x \log 3}{\log x} + \log 3 \sum_{n=2}^{x} \frac{1}{\log^2 n} ,$$

Consider the sum $\sum_{n=2}^{x} \frac{1}{\log^2 n}$.

(1.2.7) $$\sum_{n=2}^{x} \frac{1}{\log^2 n} < \frac{1}{\log^2 2} + \int_2^x \frac{dt}{\log^2 t} .$$

(1.2.8) Now consider $\phi(x) = \frac{\alpha x}{\log^2 x} - \int_2^x \frac{dt}{\log^2 t}$ where

$x \geq 2$ and $\alpha > 1$; then

(1.2.9) $$\phi'(x) = \frac{1}{\log^2 x} \left(\alpha - \frac{2\alpha}{\log x} - 1\right)$$

so that $\phi(x)$ has a minimum C_α where $C_\alpha = \phi\left(e^{\frac{2\alpha}{\alpha-1}}\right)$. Let $\alpha = 1.2$, then

(1.2.10) $$C_\alpha = \phi(e^{12}) = \frac{1.2\, e^{12}}{(12)^2} - \int_2^{e^{12}} \frac{dt}{\log^2 t}$$

$$= 1356.3 - 1396.3 = -40$$

where evaluation of the integral in (1.2.9) may be obtained by tables

of $\int_2^x \frac{dt}{\log^2 t}$, [6], or if we write

$$\int_2^x \frac{dt}{\log^2 t} = \int_2^x \frac{tdt}{t \log t} = \frac{2}{\log 2} = \frac{x}{\log x} + \int_2^x \frac{dt}{\log t},$$

by tables of $\int_2^x \frac{dt}{\log t}$, [5]. Hence we have by (1.2.8) and (1.2.10),

$$\frac{1.2x}{\log^2 x} - \int_2^x \frac{dt}{\log^2 t} \geq -40 , \tag{1.2.11}$$

i.e.

$$\int_2^x \frac{dt}{\log^2 t} \leq \frac{1.2x}{\log^2 t} + 40 .$$

Therefore by (1.2.6)

$$\begin{aligned} \pi(x) &< \frac{x \log 3}{\log x} + \log 3 \left(\frac{1}{\log^2 2} + \frac{1.2x}{\log^2 x} + 40\right) \\ &< \frac{5}{4}\,\frac{x}{\log x} \qquad \text{for} \quad x \geq 25{,}000, \end{aligned} \tag{1.2.12}$$

and a direct check of tables of $\pi(x)$ (such as those of Appel and Rosser [1]) for values of $x < 25{,}000$ concludes the proof of (1.2.4) and (1.2.5).

CHAPTER II

A THEOREM OF SYLVESTER AND SCHUR

J. J. Sylvester [14] in 1892 published a proof of the theorem that in the set of integers $n, n+1, \ldots, n+k-1$, there is a number containing a prime divisor greater than k. The theorem was later rediscovered, in 1929, by I. Schur [13]. More recent results on this theorem are an elementary proof by P. Erdos [2] and as yet unpublished proof by M. Faulkner [3] of the following theorem: let p_k be the least prime $\geq 2k$, if $n \geq p_k$ then $\binom{n}{k}$ has a prime divisor $\geq p_k$ with the exceptions $\binom{9}{2}$, $\binom{10}{3}$. In that paper the author uses inequalities for $\vartheta(x)$ and $\pi(x)$ due to Rosser and Schoenfeld [12]. A note by Leo Moser [9] states that a simple extension of Erdos' proof leads to the theorem that the product of k consecutive integers greater than k is divisible by a prime $\geq \frac{11}{10}k$ and announces a proof of the corollary appearing on page 21 of this thesis.

We will now proceed to prove by elementary means the following:

<u>Theorem 2.1.1</u> The product of k consecutive integers $n(n+1) \ldots (n+k-1)$ greater than k contains a prime divisor greater than $(3/2)k$ with the exceptions 3.4, 8.9 and $6.7.8.9.10$

We may reformulate theorem 2.1.1 as follows:
If $n \geq 2k$ then $\binom{n}{k}$ contains a prime divisor greater than $(3/2)k$ with the above exceptions.

Corollary: If $n \geq 2k$, then $\binom{n}{k}$ has a prime divisor $\geq (7/5)k$.

Lemma 2.1.1 If $\binom{n}{k}$ is divisible by a power of a prime p^{α_p}, then $p^{\alpha_p} \leq n$

Proof: The exponent β_p of a prime p in the expression $\binom{n}{k}$ is

$$(2.1.1) \qquad \beta_p = \sum_{i=1}^{[\log_p n]=\alpha_p} \left(\left[\frac{n}{p^i}\right] - \left[\frac{k}{p^i}\right] - \left[\frac{n-k}{p^i}\right] \right)$$

where $p^{\alpha_p} \leq n < p^{\alpha_p+1}$.

Now, since $\frac{a}{x} \geq \left[\frac{a}{x}\right] > \frac{a}{x} - 1$, for each i in (2.1.1) we have

$$\left[\frac{n}{p^i}\right] - \left[\frac{k}{p^i}\right] - \left[\frac{n-k}{p^i}\right] < \frac{n}{p^i} - \frac{k}{p^i} + 1 - \frac{n-k}{p^i} + 1 = 2;$$

i.e. each of the α_p terms in (2.1.1) are either 0 or 1, and hence the highest power of p dividing $\binom{n}{k}$ is α_p.

The first part of the following proof employs methods similar to those in the proof of P. Erdos [2].

(1). Let $\pi(k)$ denote the number of primes $\leq k$. Clearly for $k \geq 8$, $\pi(k) \leq (1/2)k$. Hence if $\binom{n}{k}$ had no prime factor greater than $(3/2)k$, lemma (2.1.1) implies

$$\binom{n}{k} \leq n^{(1/2)[(3/2)k]} \leq n^{(3/4)k}$$

but $$\binom{n}{k} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdots \frac{n-k+1}{1} > \left(\frac{n}{k}\right)^k$$

and therefore under assumption

$$\left(\frac{n}{k}\right)^k < n^{(3/4)k}; \quad \text{i.e.} \quad n^{1/4} < k \quad \text{which is clearly false}$$

if $k \leq n^{1/4}$. Therefore our theorem holds for $8 \leq k \leq n^{1/4}$.

Similarly, $\pi(k) < (1/3)k$ for $k > 37$ (since we may write

$$\pi(k) < k - \left[\frac{k}{2}\right] + 1 - \left[\frac{k}{3}\right] + 1 + \left[\frac{k}{6}\right] - \left[\frac{k}{5}\right] + 1 \ldots \quad),$$ and it follows

under hypothesis that $\left(\frac{n}{k}\right)^k < n^{(1/2)k}$ which is false for $k \leq n^{1/2}$, and our theorem is true for

$$37 < k \leq n^{1/2}.$$

Again by the same approach we can show $\pi(k) < (2/9)k$ for $k \geq 300$ which implies in the same manner as the above that

$$\left(\frac{n}{k}\right)^k < n^{(1/3)k}$$ and we have a contradiction for $k \leq n^{2/3}$,

i.e. our theorem holds for $300 < k \leq n^{2/3}$.

(2). Now let us consider the case when $k > n^{2/3}$. We may assume $k > 300$.

If $\binom{n}{k}$ contains no prime divisor exceeding $(3/2)k$, then by lemma 2.1.1

$$\binom{n}{k} < \prod_{p \leq (3/2)k} p \prod_{p \leq n^{1/2}} p \prod_{p \leq n^{1/3}} p \cdots, \tag{2.1.2}$$

In chapter one we proved

$$(2.1.3)\qquad 3^{n_o} > \prod_{p \le n_o} p \prod_{p \le n_o^{1/2}} p \prod_{p \le n_o^{1/3}} p \cdots .$$

Therefore, since $k > n^{2/3}$ implies $k^{\frac{1}{\ell}} \geq n^{\frac{1}{2\ell-1}}$ for $\ell \geq 2$, we have

$$(2.1.4)\qquad 3^{(3/2)k} > \prod_{p \le (3/2)k} p \prod_{p \le n^{1/3}} p \prod_{p \le n^{1/5}} p ,\cdots .$$

Now taking $n_o = n^{1/2}$ in (2.1.3), we find

$$(2.1.5)\qquad 3^{n^{1/2}} > \prod_{p \le n^{1/2}} p \prod_{p \le n^{1/4}} p \prod_{p \le n^{1/8}} p \cdots ;$$

hence, combining (2.1.2), (2.1.4) and (2.1.5), under assumption that there is no prime $> (3/2)k$ dividing $\binom{n}{k}$,

$$(2.1.6)\qquad \binom{n}{k} < \prod_{p \le (3/2)k} p \prod_{p \le n^{1/2}} p \prod_{p \le n^{1/3}} p \cdots < 3^{(3/2)k+n^{1/2}} .$$

We will now show this leads to a contradiction. Knowing $k > 300$ we can easily show by induction that $2k < 2^{\frac{k}{30}}$. Using this let us suppose that $n \geq 4k$, still under the assumption that no prime exceeding $(3/2)k$ divides $\binom{n}{k}$, then

$$(2.1.7)\qquad 3^{(3/2)k+n^{1/2}} > \binom{4k}{k} = \binom{2k}{k}\frac{4k}{2k}\cdot\frac{4k-1}{2k-1}\cdots\frac{3k+1}{k+1} > \frac{4^k \cdot 2^k}{2k} > \frac{2^{3k}}{2^{\frac{k}{30}}}$$

since $\binom{2k}{k} \geq \frac{4^k}{2k}$.

Hence

$$(2.1.8) \quad \left((3/2)k + n^{1/2}\right) \log 3 > 2^{\frac{29}{30}k} \log 2 ,$$

i.e. $\quad n^{1/2} \log 3 > n^{2/3}\left(2^{\frac{29}{30}} \log 2 - 3/2 \log 3\right),$

which implies $n^{1/6} < 2.9$, which yields a contradiction for $n \geq 400$.

(3) Next suppose $3k \leq n < 4k$. Then, as in (2), we have

$$(2.1.9) \qquad 3^{(3/2)k+n^{1/2}} > \binom{3k}{k} = \frac{3k}{2k} \cdot \frac{3k-1}{2k-1} \cdots \frac{2k+1}{k+1} \binom{2k}{k} > \left(\frac{3}{2}\right)^k 2^{1\frac{29}{30}k}$$

which implies

$$(2.1.10) \qquad \left((3/2)k + n^{1/2}\right) \log 3 > k\left(\log 3 - \log 2 + 1\frac{29}{30} \log 2\right)$$

and therefore since $n < 4k$,

$$\left((1/2)k^{1/2} + 2\right) \log 3 > k^{1/2} \frac{29}{30} \log 2 ,$$

which is false for $k > 20$ and our theorem holds for $n \geq 80$.

Lemma 2.1.2 There is a prime between $3n$ and $4n$ for $n > 1$.

Proof: Assume the contrary. Consider the binomial coefficient $\binom{4n}{n}$. It is easy to see from the structure of $\binom{4n}{n}$ that no prime p, such that $2n < p \leq 3n$, occurs in $\binom{4n}{n}$. Thus under hypothesis there is no prime between $2n$ and $4n$ occuring in the binomial coefficient.

If α_p is such that $p^{\alpha_p} \leq 2n < p^{\alpha_p + 1}$, then since

(2.1.11) $\quad p \geq 2, \quad 4n < p^{\alpha_p + 2}$. This may be seen if we consider

$$p^{\alpha_p+2} = pp^{\alpha_p+1} > p2n \geq 4n .$$

If α_p is the exponent of a prime p in $\binom{4n}{n}$ then

$$\alpha_p = \sum_{i=1}^{[\log_p 4n]} \left(\left[\frac{4n}{p^i}\right] - \left[\frac{3n}{p^i}\right] - \left[\frac{n}{p^i}\right] \right)$$

If we now write $n = \beta_{p,i}\, p^i + \gamma_{p,i}$, $0 \leq \gamma_{p,i} < p^i$ and analyze the term under summation for fractional parts of $\gamma_{p,i}$ in the ranges

(i) $0 \leq \gamma_{p,i} < 1/4\ p^i$, (ii) $1/4\ p^i \leq \gamma_{p,i} < 1/3\ p^i$

(iii) $1/3\ p^i \leq \gamma_{p,i} < 1/2\ p^i$, (iv) $1/2\ p^i \leq \gamma_{p,i} < 2/3\ p^i$

(v) $2/3\ p^i \leq \gamma_{p,i} < 3/4\ p^i$, and (vi) $3/4\ p^i \leq \gamma_{p,i} < p^i$

we find

$$\left[\frac{4n}{p^i}\right] - \left[\frac{3n}{p^i}\right] - \left[\frac{n}{p^i}\right] = 0,\ 1,\ 0,\ 1,\ 0 \text{ and } 1 \text{ respectively .}$$

Thus a prime p occuring to a power greater than one in $\binom{4n}{n}$ must satisfy $p \leq (4n)^{1/2}$. Therefore under hypothesis and by (2.1.11), we have

(2.1.12) $$\binom{4n}{n} \leq \prod_{p^{\alpha} \leq 2n} p \prod_{p \leq (4n)^{1/2}} p ,$$

i.e. $$\binom{4n}{n} < 3^{2n+(4n)^{1/2}} .$$

On the other hand we can prove by induction that $\binom{4n}{n} > \left(\frac{4^4}{3^3}\right)^n \frac{1}{4n}$; hence under hypothesis

(2.1.13) $$\left(\frac{4^4}{3^3}\right)^n \frac{1}{4n} < 3^{2n+(4n)^{1/2}} ,$$

which is false for $n \geq 2200$, and a straight-forward check of a table of primes for $1 \leq n < 2200$ concludes the proof of lemma 2.1.2.

Finally if we consider the case when $2k \leq n < 3k$, our conclusion follows by lemma 2.1.2 for $k > 4$ since the greatest integer less than or equal to $2k$ and divisible by 3 is greater than $(3/2)k$.

Thus our theorem holds for $k \geq 8$ with a finite number of exceptions which may be checked by a table of primes.

Consider the case $k = 5$, we want to show that one of the numbers, $n, n-1, \ldots, n-4$ where $n - 4 > 5$ is divisible by a prime greater than $3/2 \cdot 5$, i.e. divisible by a prime ≥ 11. Assume the contrary and consider the binomial coefficient $\binom{n}{5}$. By lemma 2.1.1 we have that the greatest contribution of any prime p to $\binom{n}{5}$ is at most n; hence under assumption

$$\binom{n}{5} < n^{\pi(3/2 \cdot 5)} = n^4 ,$$

i.e. $$\frac{n(n-1)\ldots(n-4)}{5.4.3.2.1} < n^4$$

which is certainly false for say $n \geq 100$, and a check of a table of primes for values of n less than 100 reveals one exception to our theorem, i.e. the example 6.7.8.9.10 which has no prime divisor > 7. We may treat the case $k = 4$ in exactly the same manner, and no exceptions to our theorem occur.

The cases $k = 6$ and $k = 7$ now follow from the case $k = 5$ since we have proven any five consecutive numbers greater than five contain a prime ≥ 11. Then since $3/2 \cdot 6 < 11$, and $3/2 \cdot 7 < 11$ our theorem holds without exception for $k = 6$ and $k = 7$.

For $k = 3$, consider the integers n, $n+1$, $n+2$, $n > 3$. If $n \equiv 0(3)$, then either n or $n+1$ is divisible by a prime greater than 3 since $(n, n+1) = 1$ and $n > 3$. The case $n+2 \equiv 0(3)$, is identical. If $n+1 \equiv 0(3)$, since consecutive integers are relatively prime, the only time when n and $n+2$ are not divisible by a prime greater than 3 is when both n and $n+2$ are powers of 2, which implies $n = 2$. Therefore our theorem holds for $k = 3$.

When $k = 2$, by the same approach we have the exceptions 3.4 and 8.9 and otherwise the theorem is valid, and the case $k = 1$ is trivially true.

The exception $\binom{10}{5}$ proves the corollary to theorem 2.1.1, i.e. that $7/5$ is the 'best possible' constant c such that $\binom{n}{k}$ is divisible by a prime $\geq ck$ for $n \geq 2k$.

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